International Journal of Theoretical Physics, Vol. 43, Nos. 7/8, August 2004 (© 2004)

A Note on Strongly Closed Subspaces in an Inner Product Space

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We study some weak states on the orthocomplemented lattice $F(S)$ of all strongly closed subspaces in an inner product space. We show that $F(S)$ always possesses two-valued weak states, and we derive a characterization of (topological) completeness of S in terms of two-valued weak states. We also prove an extension theorem for weak states.

KEY WORDS: inner product space; lattice of strongly closed subspaces; (weak) state.

1. INTRODUCTION

Suppose that S is an inner product space over real numbers. Let $\langle .\,, . \rangle$ denote the inner product on S. Following Maeda and Maeda (1970), let us consider the subspaces of S which coincide with their double polar, i.e., let us set $F(S) = \{A \subset S \mid A = (A^{\perp})^{\perp}\}$, where $A^{\perp} = \{b \in S \mid \langle a, b\rangle = 0$ for each $a \in A\}$. Let us call an element of $F(S)$ a strongly closed subspace of S. As shown in Maeda and Maeda (1970), $F(S)$ endowed with inclusion and polar orthocomplementation is a complete orthocomplemented lattice which enjoys rather interesting properties related to the theory of quantum logics (see Amemiya and Araki, 1966; Dvurečenskij, 1993; Dvurečenskij and Pulmannová, 1988; Hamhalter and Pták, 1987; Pták, 1988; Pták and Weber, 1998).

It is an open question (Pták, 1988) whether or not $F(S)$ possesses a finitely additive state (the absence of the σ-additive states for S incomplete was shown in Hamhalter and Pták (1987). It seems instrumental in connection with the latter problem to study weak states first (a weak state is a certain natural generalization of state, see Definition 1). In this note we concern ourselves with weak states, obtaining the results stated in the abstract.

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0020-7748/04/0800-1737/0

2. RESULTS

Let $F(S)$ denote the collection of all strongly closed subspaces of an inner product space. We shall mainly deal with the following notion of weak state on $F(S)$.

Definition 1. A weak state on $F(S)$ is a mapping $w : F(S) \to \langle 0, 1\rangle$ which satisfies the following properties

1. $w(\{0\}) = 0$,
2. $w(M) + w(M^{\perp}) = 1$ for any $M \in F(S)$,
3. if $M \subset N$, $M, N \in F(S)$, then $w(M) \leq w(N)$.

Let us denote the set of all weak states (resp. the set of all two-valued weak states) on $F(S)$ by $S^{w}(F(S))$ (resp. $\mathcal{S}_2^{w}(F(S))$).

In our first result we show that $\mathcal{S}_2^{w}(F(S))$ is always considerably rich.

Theorem 1. *Let $M \in F(S)$, $M \neq \{0\}$. Then there is a weak state, w, on $F(S)$ such that $w(M) = 1$.*

Proof: We first prove a lemma. □

Lemma 2. *Let $\mathcal{P} \subset F(S)$ be a collection of subspaces which satisfies the following properties:*

(i) if $M \in \mathcal{P}$. then $M^{\perp} \notin \mathcal{P}$,
(ii) if $M \in \mathcal{P}$ and $M \subset N$, $N \in F(S)$, then $N \in \mathcal{P}$,
(iii) $\{0\} \notin \mathcal{P}$.

Then there is a collection $\mathcal{Q}$, $\mathcal{P} \subset \mathcal{Q}$ which satisfies all the properties (i), (ii), (iii), and which enjoys the property of selectivity:
If $K \in F(S)$, then $card(\{K, K^{\perp}\} \cap \mathcal{Q}) = 1$.

Proof: The proof follows the standard pattern of "maximality" proofs. Let us denote by $\mathcal{R}$ the set of all collections $\mathcal{P}' \subset F(S)$ with the properties (i), (ii), (iii) and with the property $\mathcal{P} \subset \mathcal{P}'$. Let us order $\mathcal{R}$ by inclusion. Suppose that C is a chain in $\mathcal{R}$. Denote by $\mathcal{D}$ its union. Thus, $\mathcal{D} = \bigcup_{P' \in C} P'$. Then $\mathcal{D}$ again satisfies (i), (ii), (iii).

Indeed, if $M, M^{\perp} \in \mathcal{D}$, then $M \in \mathcal{P}_1$ and $M^{\perp} \in \mathcal{P}_2$ for some $\mathcal{P}_1, \mathcal{P}_2 \in C$. If $\mathcal{P}_1 \subset \mathcal{P}_2$, then $M \in \mathcal{P}_2$ while if $\mathcal{P}_2 \subset \mathcal{P}_1$, then $M^{\perp} \in \mathcal{P}_1$ – a contradiction. Further, if $M \in \mathcal{D}$ and $M \subset N$, $N \in F(S)$, then $M \in \mathcal{P}$ for some $\mathcal{P} \in C$. Hence, $N \in \mathcal{P}$ which gives $N \in \mathcal{D}$. The property (iii) is evident for $\mathcal{D}$.

Let us proceed by Zorn's lemma: There exists a maximal element, $\mathcal{Q}$, which satisfies all the conditions (i), (ii), (iii). Suppose that $K \notin \mathcal{Q}$. If $L \not\subset K$ for every

$L \in \mathcal{Q}$, then by the possibility to add K and all $T \supset K$ to $\mathcal{Q}$, we infer that $\mathcal{Q}$ is not maximal. Thus, there is some $N \in \mathcal{Q}$ such that $K \perp N$ and therefore $N \subset K^\perp$. It follows that $K^\perp \in \mathcal{Q}$ and the proof of Lemma 2 is complete. □

Returning to the proof of Theorem 1, let us start off with the collection $\mathcal{M} = \{N \in F(S),\ M \subset N\}$. This $\mathcal{M}$ satisfies (i), (ii), (iii) and, by Lemma 2, $\mathcal{M}$ can be enlarged to a selective one, $\mathcal{Q}$. Letting then $w(P) = 1$ if $P \in \mathcal{Q}$, we have the desired two-valued weak state.

The abundance of two-valued weak states allows us in turn to prove the following two results. (It should be observed that there is no analogue for the (ordinary) states, see Dvurečenskij (1993)– there are no two-valued states on $F(S)$.)

The first result which follows presents a certain completeness criterion.

Theorem 3. *Let S be an inner product space. Then S is Hilbert if, and only if, for each two-valued weak state $w \in \mathcal{S}_2^w(F(S))$ there is a (topologically) complete subspace $C \subset S$ such that $w(C) = 1$.*

Proof: If S is Hilbert, then the statement is evident. Suppose that S is incomplete and consider the subcollection $\mathcal{P} \subset F(S)$ determined as follows (by a co-complete space we mean a space the polar of which is complete)

$$\mathcal{P} = \{M \in F(S) \mid \text{there is a cocomplete } N \text{ such that} N \subset M\}.$$

It is easily seen that we can write $\mathcal{P} = \{M \in F(S) \mid M \text{ is co-complete}\}$. Obviously, $\mathcal{P}$ satisfies (i), (ii), and (iii) of Lemma 2. Indeed, (ii) follows from the definition of $\mathcal{P}$ and (i) and (iii) immediately follow from the fact that S is not complete. Using Zorn's lemma, we can find a maximal element, $\mathcal{Q}$, containing $\mathcal{P}$. This $\mathcal{Q}$ defines a two-valued state, $w \in \mathcal{S}_2^w(F(S))$ as demonstrated above. Obviously, $w(C) = 0$ for every complete $C \subset S$ (evidently, each complete subspace belongs to $F(S)$). This proves Theorem 3. □

The next result shows that each state of a Boolean subalgebra of $F(S)$ extends over $F(S)$ to a weak state. In other words, it is shown that the restriction of weak states on an arbitrary Boolean subalgebra of $F(S)$ is a surjection onto all states of the subalgebra. By a Boolean subalgebra we naturally mean a complemented sublattice which is a Boolean algebra in its own right. (Recall that by a state on a Boolean algebra B we mean a mapping $s\colon B \to \langle 0, 1\rangle$ which satisfies the three requirements of weak state and which, in addition, is additive: $s(a \vee b) = s(a) + s(b)$ provided $a \le b^\perp$.)

We would need the following strengthening of Theorem 1: If $\mathcal{B}$ is a Boolean subalgebra of $F(S)$ and if $M \in F(S)$, $M \neq \{0\}$, then there is a weak two-valued state w on $F(S)$ such that $w(M) = 1$ and, moreover, w is a state on $\mathcal{B}$. Let us call a weak state which is additive on $\mathcal{B}$ a $\mathcal{B}$-additive weak state. (We omit the proof of the above strengthening of Theorem 1 since it can be obtained by a straightforward generalization of Lemma 2.)

Theorem 4. *Let S be an inner product space and let $\mathcal{B}$ be a Boolean subalgebra of $F(S)$. Let s be a state on $\mathcal{B}$. Then there is a weak state $w \in \mathcal{S}^w(F(S))$ such that $w(A) = s(A)$ for each $A \in \mathcal{B}$.*

Proof: Let us first observe that $\mathcal{S}^w(F(S))$ forms a convex subset of $\langle 0, 1\rangle^{F(S)}$ and that $\mathcal{S}^w(F(S))$ is compact in $\langle 0, 1\rangle^{F(S)}$ when understood with the Tychonoff (=pointwise) topology. We will now use the adequate analogue of the technique in Pták (1985) and Tkadlec (1991). Let us consider the set $\mathcal{P}$ of all partitions of $\mathcal{B}$ (by a partition, P, of B we mean a finite mutually orthogonal collection $\{P_1, P_2, \ldots, P_n\} \subset B$ such that $\bigvee_{i=1}^n P_i = 1$ in B). Obviously, $\mathcal{P}$ is a directed set when ordered by the refinement relation. Let us set

$$\mathcal{F}_P = \{w \in \mathcal{S}^w(F(S)) \mid w \text{ is } \mathcal{B}\text{-additive and } w(P_i) = s(P_i) \text{ for each } P_i \in P\}.$$

Let us show that $\mathcal{F} = \{\mathcal{F}_P \mid P \in \mathcal{P}\}$ is a filter base consisting of closed subsets of $\mathcal{S}^w(F(S))$. Since $\mathcal{S}^w(F(S))$ is endowed with the pointwise topology, the fact that every $\mathcal{F}$ is closed is obvious. To show that each $\mathcal{F}_P$ is nonvoid, take two partitions, $P, R \in \mathcal{P}$, $P = \{P_1, \ldots, P_m\}$, $R = \{R_1, \ldots, R_n\}$ and form the intersection partition $Q = P \cap R = \{P_i \cap R_j \mid i \leq m, j \leq n\}$. Write $Q = \{Q_1, \ldots, Q_d\}$ and, for each non-zero Q_i, find a (two-valued) weak $\mathcal{B}$-additive state $w_{Q_i} \in \mathcal{S}_2^w(F(S))$ such that $w_{Q_i}(Q_i) = 1$. Then the mapping $w_Q = \sum_{i \leq d} s(Q_i)\, w_{Q_i}$ is a convex combination of weak states on $F(S)$ and therefore $w_Q \in \mathcal{S}^w(F(S))$. Since Q refines both P and R, we easily see that $w_Q \in \mathcal{F}_P \cap \mathcal{F}_R$. Thus, $\mathcal{F}$ is a filter base.

The rest uses compactness in the standard manner. Having $\mathcal{S}^w(F(S))$ compact, there is an element, w, such that $w \in \bigcap_{P \in \mathcal{P}} \mathcal{F}_P$. By the definition of $\mathcal{F}_P$ we immediately see that $w(A) = s(A)$ for each $A \in \mathcal{B}$. The proof is complete. $\square$

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